stichting
mathematisch
centrum

MC

AFDELING NUMERIEKE WISKUNDE NW 42/77 MEI
(DEPARTMENT OF NUMERICAL MATHEMATICS)

P.J. VAN DER HOUWEN

ON THE NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

I STABILITY

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 65R05

On the numerical solution of Volterra integral equations of the second kind
I Stability

by

P.J. van der Houwen

ABSTRACT

The main purpose of this paper is to analyse the stability of algorithms for non-linear Volterra integral equations of the second kind. In particular, Runge-Kutta type methods are studied.

KEY WORDS & PHRASES: *Integral equations, Volterra, stability, Runge-Kutta type formulas*

CONTENTS

1. INTRODUCTION

In this paper the stability behaviour is considered of linear multistep methods and single-step methods of Runge-Kutta type when these methods are applied to Volterra integral equations of the second kind. In particular, for Runge-Kutta methods the stability problem seems to be a hardly investigated area, presumably because Runge-Kutta methods are considered more as starting formulas for multistep methods than as independent integration formulas. It turns out, however, that Runge-Kutta type methods allow relatively large regions of stability and therefore may be advantageous in cases where the kernel function of the integral equation has a large Lipschitz constant. Hence, we concentrate our considerations on Runge-Kutta methods, although we also give the corresponding theory for multistep methods. In section 2 the general formulas are introduced and a modification of these formulas is discussed, which can be characterized by the property that they reduce to the common linear multistep and Runge-Kutta methods for ordinary differential equations in those cases where the integral equation is equivalent to an ordinary initial value problem. In section 3 the consistency conditions are derived and in section 4 a convergence theorem is given, both for the "usual" and modified form of the multistep and Runge-Kutta schemes. It appears that the modified formulas generally have a lower order of accuracy, however, as will be shown in section 5, the stability analysis is easier and, as we will report in a forthcoming report, the stability regions seem to be larger. Numerical experiments will be published in the near future.

2. DERIVATION OF A COMPUTATIONAL SCHEME

Volterra integral equations of the second kind may be written in the form

$$f(x) = F_n(x) + \int_{x_n}^{x} K(x,\xi,f(\xi))d\xi, \tag{2.1}$$

where

$$F_n(x) = g(x) + \int_{x_0}^{x_n} K(x,\xi,f(\xi))d\xi.$$

With respect to the point x_n, the first term $F_n(x)$ may be interpreted as the "past" and the second term as the "future" of the integral equation. When approximations f_j to $f(x_j)$, $j = 0,1,\dots,n$ are obtained, we may approximate $F_n(x)$ for arbitrary x-values by applying some quadrature formula and by replacing $f(x_j)$ by f_j, i.e.

$$(2.2) \qquad F_n(x) \cong \tilde{F}_n(x) = g(x) + \sum_{j=0}^{n} w_{nj}\, K(x,x_j,f_j).$$

In order to derive a formula for the numerical approximation f_{n+1}, we consider the following formula for $f(x_{n+1})$:

$$(2.3) \qquad f(x_{n+1}) = F_n(x_{n+1}) + \int_{x_n}^{x_{n+1}} K(x_{n+1},\xi,f(\xi))d\xi.$$

By replacing $F_n(x_{n+1})$ with some approximation $\tilde{F}_n(x_{n+1})$ and the integral by a numerical quadrature formula, we obtain a formula of the type

$$(2.3') \qquad f_{n+1} = \tilde{F}_n(x_{n+1}) + \Phi_n(K(x,\xi,\tilde{f}(\xi))),$$

where $\tilde{f}$ represents some interpolating function through the values f_j, $j = n+1,n,\dots;\Phi_n$ denotes some approximation to the integral in formula (2.3). Two cases will be considered: firstly, the integral will be replaced by a formula using non-step points in a way as is done in Runge-Kutta formulas for differential equations, and secondly, we replace the integral by a linear multistep formula. Both approaches are well-known in the literature. The Runge-Kutta approach may be found in e.g. LAUDET and OULES [7], POUZET [10], DAY [2], BELTJUKOV [1] and DE HOOG and WEISS [3]. Multistep methods were considered by e.g. KOBAYASI [6], LINZ [8], NOBLE [9] and GAREY [4].

2.1 SINGLE-STEP METHODS

Similar to Runge-Kutta methods for differential equations we may define the scheme

$$(2.4) \qquad f_{n+1}^{(0)} = f_n, f_{n+1}^{(j)} = \tilde{F}_n(x_n+\mu_j h_n) + h_n \sum_{\ell=0}^{m} \lambda_{j\ell}\, K(x_n+\theta_{j\ell}h_n, x_n+\nu_{j\ell}h_n, f_{n+1}^{(\ell)}),$$
$$f_{n+1} = f_{n+1}^{(m)} \qquad j = 1,2,\dots,m.$$

Here, h_n denotes the step length $x_{n+1} - x_n$ and $\tilde{F}_n(x_n+\mu_j h_n)$ denotes some approximation to $F_n(x_n+\mu_j h_n)$. After m iterations the approximant $f_{n+1}^{(m)}$ is taken as the final approximation f_{n+1} to $f(x_{n+1})$. The parameters μ_j, $\lambda_{j\ell}$, $\theta_{j\ell}$ and $\nu_{j\ell}$ are determined by consistency and stability conditions. We observe that the computational effort per step of formula (2.4) can be reduced by trying to give μ_j either the value 0 (since $F_n(x_n) = f(x_n)$) or to give μ_j a value independent of j.

It may be interesting to consider the class of integral equations with

$$\frac{\partial K}{\partial x} = \frac{dg}{dx} = 0. \tag{2.5}$$

Equation (2.1) then reduces to a *differential equation* of the form

$$\frac{df}{dx} = K(x,x,f) = K^*(x,f) \tag{2.1'}$$

The corresponding numerical scheme reduces to (note that $F_n(x)$ no longer depends on x)

$$\begin{aligned} f_{n+1}^{(0)} &= f_n \\ f_{n+1}^{(j)} &= \tilde{F}_n + h_n \sum_{\ell=0}^{m} \lambda_{j\ell} K^*(x_n+\nu_{j\ell}h_n, f_{n+1}^{(\ell)}), \quad j = 1,2,\ldots.m, \\ f_{n+1} &= f_{n+1}^{(m)}, \end{aligned} \tag{2.4'}$$

whereas direct application of the general m-point Runge-Kutta to equations (2.1') yields a scheme of the type

$$\begin{aligned} f_{n+1}^{(0)} &= f_n \\ f_{n+1}^{(j)} &= f_n + h_n \sum_{\ell=0}^{m} \lambda_{j\ell} K^*(x_n+\nu_\ell h_n, f_{n+1}^{(\ell)}), \quad j = 1,2,\ldots m \\ f_{n+1} &= f_{n+1}^{(m)}. \end{aligned} \tag{2.4''}$$

A comparison of (2.4') and (2.4") suggest to define $\tilde{F}_n(x)$ is such a way that $\tilde{F}_n(x) = f_n$ for all integral equations satisfying (2.5). In that case all theory derived for Runge-Kutta methods also applies to this special class of integral equations (we will call these equations *test equations*). Let us consider formula (2.2) more closely by writing it in the form

$$(2.6)\qquad \tilde{F}_n(x) = \tilde{F}_{n+1}(x_n) + g(x) - g(x_n) + \sum_{j=0}^{n} w_{nj}\, K(x,x_j,f_j) - \sum_{j=0}^{n-1} w_{n-1j}\, K(x_n,x_j,f_j).$$

Using representation (2.3') this formula transform into

$$(2.7)\qquad \tilde{F}_n(x) = f_n - \Phi_{n-1}(K(x,\xi,\tilde{f})) + g(x) - g(x_n) + \sum_{j=0}^{n} [w_{nj}\, K(x,x_j,f_i) - w_{n-1j}\, K(x_n,x_j,f_j)],$$

where the weights w_{ij} are assumed to be zero when $i < j$.

Generally, (2.7) will not reduce to the equation $\tilde{F}_n(x) = f_n$ when applied to the test equation. However, when we define $\tilde{F}_n(x)$ by the formula

$$(2.2')\qquad \tilde{F}_n(x) = f_n + g(x) - g(x_n) + \sum_{j=0}^{n} w_{nj}[K(x,x_j,f_j) - K(x_n,x_j,f_j)],$$

it is still a consistent approximation to $F_n(x)$, while it reduces to $\tilde{F}_n(x) = f_n$ for the class of test equations. Unfortunately, formula (2.2') may reduce the order of accuracy by one as will be shown in section 4. In the following we shall call a formula, in which $\tilde{F}_n(x)$ is evaluated according to (2.2') instead of using the direct quadrature formula (2.2), a *modified* integration formula.

2.2 MULTISTEP METHODS

We replace the integral in formula (2.3) by a linear k-step formula to obtain

$$(2.8)\qquad f_{n+1} = \tilde{F}_n(x_{n+1}) + h_n \sum_{\ell=0}^{k} b_{n\ell}\, K(x_{n+1},x_{n+1-\ell},f_{n+1-\ell}).$$

The parameters $b_{n\ell}$ are determined by consistency and stability conditions.

Let us apply this scheme to integral equations satisfying (2.5) and suppose that $\tilde{F}_n(x)$ is defined by (2.2'). We then have

$$(2.8') \qquad f_{n+1} = f_n + h_n \sum_{\ell=0}^{k} b_{n\ell}\, K(x_{n+1-\ell}, f_{n+1-\ell})$$

which is identical to the well-known Adams formula for ordinary differential equations. Thus, just as for Runge-Kutta formulas, we see that the modified forms reduce to formulas known for ordinary differential equations when they are applied to the class of test equations.

It should be observed that the weights w_{ij}, $j = 0,1,\dots,i$, $i = 1,2,\dots,n$ used in the computation of $\tilde{F}_n(x_{n+1})$ are not necessarily related to the parameters $b_{\nu\ell}$, $\ell = 0,1,\dots,k$; $\nu = 1,2,\dots,n$. Usually, however, one has

$$(2.9) \qquad h_n b_{n,\ell} = w_{n+1,n+1-\ell} = w_{n,n+1-\ell},$$

where $w_{n,n+1}$ is assumed to be zero. The numerical algorithm then simply reads

$$(2.10) \qquad f_{n+1} = g(x_{n+1}) + \sum_{j=0}^{n+1} w_{n+1,j}\, K(x_{n+1}, x_j, f_j).$$

3. CONSISTENCY CONDITIONS

3.1 SINGLE STEP METHODS

Instead of appproximating the integral in (2.3) by a direct quadrature rule based on non-step points (cf. DE HOOG and WEISS [3]), we try to determine the Runge-Kutta parameters in (2.4) along the same lines as is done in ordinary differential equations, i.e. by deriving and solving the *consistency conditions* (cf. BELTJUKOV [1]). Scheme (2.4) will be called consistent of order p when

$$f_{n+1} - f(x_{n+1}) = \mathcal{O}(h_n^{p+1}) \text{ as } h_n \to 0,$$

where f_{n+1} is assumed to be the result of formula (2.4) when applied with $f_j = f(x_j)$ and $\tilde{F}_j(x) = F_j(x)$, $j = 0,1,\ldots,n$ (in analogy to the terminology in ordinary differential equations called "the localizing assumption"). Consistency conditions can be derived by expanding f_{n+1} and $f(x_{n+1})$ in a Taylor series about the point x_n. For f_{n+1} we may write

$$
\begin{aligned}
(3.1)\qquad f_{n+1} = {} & F_n(x_n+\mu_m h_n) + h_n \sum_{\ell=0}^{m} \lambda_{m\ell} \Big[K(x_n,x_n,f(x_n)) + \\
& + \theta_{m\ell} h_n \frac{\partial K}{\partial x} + \nu_{m\ell} h_n \frac{\partial K}{\partial \xi} + \left(f_{n+1}^{(\ell)} - f(x_n)\right) \frac{\partial K}{\partial f} + \tfrac{1}{2}\theta_{m\ell}^2 h_n^2 \frac{\partial^2 K}{\partial x^2} + \\
& + \theta_{m\ell} h_n \left(f_{n+1}^{(\ell)} - f(x_n)\right) \frac{\partial^2 K}{\partial x \partial f} + \tfrac{1}{2}\nu_{m\ell}^2 h_n^2 \frac{\partial^2 K}{\partial \xi^2} + \\
& + \nu_{m\ell} h_n \left(f_{n+1}^{(\ell)} - f(x_n)\right) \frac{\partial^2 K}{\partial \xi \partial f} + \tfrac{1}{2} \left(f_{n+1}^{(\ell)} - f(x_n)\right)^2 \frac{\partial^2 K}{\partial f^2} + \\
& + \theta_{m\ell}\, \nu_{m\ell} h_n^2 \frac{\partial^2 K}{\partial x \partial \xi} + \mathcal{O}(h_n^3)\Big].
\end{aligned}
$$

Here, all partial derivatives of K are evaluated at the point $(x_n,x_n,f(x_n))$. In order to write f_{n+1} in a power series of h_n we have to expand $f_{n+1}^{(\ell)}$ in a power series of h_n:

$$
\begin{aligned}
f_{n+1}^{(\ell)} = {} & F_n(x_n) + \mu_\ell h_n F_n'(x_n) + \tfrac{1}{2}\mu_\ell^2 h_n^2 F_n''(x_n) + \\
& + h_n \sum_{i=0}^{m} \lambda_{\ell i} \Big[K + \theta_{\ell i} h_n \frac{\partial K}{\partial x} + \nu_{\ell i} h_n \frac{\partial K}{\partial \xi} + \\
& \qquad + \left(f_{n+1}^{(i)} - f(x_n)\right) \frac{\partial K}{\partial f} + \mathcal{O}(h_n^2)\Big] = \\
= {} & f(x_n) + h_n \Big[\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K\Big] + \\
& + h_n^2 \Big[\tfrac{1}{2}\mu_\ell^2 F_n''(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} \Big(\theta_{\ell i} \frac{\partial K}{\partial x} + \nu_{\ell i} \frac{\partial K}{\partial \xi} + \\
& \qquad + (\mu_i F_n'(x_n) + \sum_{k=0}^{m} \lambda_{ik} K) \frac{\partial K}{\partial f}\Big)\Big] + \mathcal{O}(h_n^3), \\
& \qquad \ell = 0,1,\ldots,m,
\end{aligned}
$$

where μ_0 and λ_{0i} are assumed to be zero ($f_{n+1}^{(0)} = f_n = f(x_n)$).

Substitution in (3.1) leads to the power series

$$\begin{aligned}
(3.2)\qquad f_{n+1} = {} & F_n(x_n + \mu_m h_n) + h_n \sum_{\ell=0}^{m} \lambda_{m\ell} K + \\
& + h_n^2 \sum_{\ell=0}^{m} \lambda_{m\ell} \left\{ \theta_{m\ell} \frac{\partial K}{\partial x} + \nu_{m\ell} \frac{\partial K}{\partial \xi} + \left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K \right) \frac{\partial K}{\partial f} \right\} + \\
& + h_n^3 \sum_{\ell=0}^{m} \lambda_{m\ell} \left\{ \tfrac{1}{2}\mu_\ell^2 F_n''(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} \left(\theta_{\ell i} \frac{\partial K}{\partial x} + \nu_{\ell i} \frac{\partial K}{\partial \xi} + \right.\right. \\
& \left. + (\mu_i F_n'(x_n) + \sum_{k=0}^{m} \lambda_{ik} K \right) \frac{\partial K}{\partial f} \Bigg) \frac{\partial K}{\partial f} + \\
& + \tfrac{1}{2}\theta_{m\ell}^2 \frac{\partial^2 K}{\partial x^2} + \theta_{m\ell} \left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K \right) \frac{\partial^2 K}{\partial x \partial f} \\
& + \tfrac{1}{2}\nu_{m\ell}^2 \frac{\partial^2 K}{\partial \xi^2} + \nu_{m\ell} \left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K \right) \frac{\partial^2 K}{\partial \xi \partial f} \\
& + \tfrac{1}{2} \left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K \right)^2 \frac{\partial^2 K}{\partial f^2} + \\
& + \theta_{m\ell} \nu_{m\ell} \frac{\partial^2 K}{\partial x \partial \xi} \Bigg\} + \\
& + 0(h_n^4).
\end{aligned}$$

On the other hand, it follows from (2.3) that

$$\begin{aligned}
f(x_{n+1}) = {} & F_n(x_{n+1}) + \int_{x_n}^{x_{n+1}} \Big[K(x_n, x_n, f(x_n)) + (\xi - x_n) \frac{\partial K}{\partial \xi} + \\
& (x - x_n) \frac{\partial K}{\partial x} + (f(\xi) - f(x_n)) \frac{\partial K}{\partial f} + \tfrac{1}{2}(\xi - x_n)^2 \frac{\partial^2 K}{\partial \xi^2} + \tfrac{1}{2}(x - x_n)^2 \frac{\partial^2 K}{\partial x^2} + \\
& + (\xi - x_n)(f(\xi) - f(x_n)) \frac{\partial^2 K}{\partial \xi \partial f} + \tfrac{1}{2}(f(\xi) - f(x_n))^2 \frac{\partial^2 K}{\partial f^2} + \\
& + (x - x_n)(f(\xi) - f(x_n)) \frac{\partial^2 K}{\partial x \partial f} + (x - x_n)(\xi - x_n) \frac{\partial^2 K}{\partial x \partial \xi} + 0(h_n^3) \Big].
\end{aligned}$$

Expansion of $f(\xi)$ about the point x_n and integration yields the series

$$(3.3) \qquad f(x_{n+1}) = F_n(x_{n+1}) + h_n K +$$
$$+ \tfrac{1}{2}h_n^2 \left[2 \frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi} + f'(x_n) \frac{\partial K}{\partial f}\right] +$$
$$+ \frac{1}{6} h_n^3 \left[f''(x_n) \frac{\partial K}{\partial f} + 3 \frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial \xi^2} +\right.$$
$$\left. + (f'(x_n))^2 \frac{\partial^2 K}{\partial f^2} + 3 \frac{\partial^2 K}{\partial x \partial \xi} + 2f'(x_n) \frac{\partial^2 K}{\partial \xi \partial f} + 3f'(x_n) \frac{\partial^2 K}{\partial x \partial f}\right]$$
$$+ O(h_n^4).$$

Hence,

$$(3.4) \qquad f_{n+1} - f(x_{n+1}) = \left[F_n(x_n + \mu_m h_n) - F_n(x_{n+1})\right] +$$
$$+ h_n \left[\sum_{\ell=0}^{m} \lambda_{m\ell} - 1\right] K +$$
$$+ h_n^2 \left[\left(\sum_{\ell=0}^{m} \lambda_{m\ell}\nu_{m\ell} - \tfrac{1}{2}\right) \frac{\partial K}{\partial \xi} + \left(\sum_{\ell=0}^{m} \lambda_{m\ell}\theta_{m\ell} - 1\right) \frac{\partial K}{\partial x} +\right.$$
$$\left. + \left(\sum_{\ell=0}^{m} \lambda_{m\ell}\left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K\right) - \tfrac{1}{2}f'(x_n)\right) \frac{\partial K}{\partial f}\right] +$$
$$+ h_n^3 \left[\left(\sum_{\ell=0}^{m} \lambda_{m\ell}\left(\tfrac{1}{2}\mu_\ell^2 F_n''(x_n) + \sum_{i=0}^{m} \lambda_{\ell i}\left(\nu_{\ell i} \frac{\partial K}{\partial \xi} + \theta_{\ell i} \frac{\partial K}{\partial x}\right) +\right.\right.\right.$$
$$\left.\left. + \sum_{i=0}^{\ell} \lambda_{\ell i}\left(\mu_i F_n'(x_n) + \sum_{k=0}^{m} \lambda_{ik} K\right) \frac{\partial K}{\partial f}\right) - \frac{1}{6} f''(x_n)\right) \frac{\partial K}{\partial f} +$$
$$+ \left(\sum_{\ell=0}^{m} \tfrac{1}{2}\lambda_{m\ell}\nu_{m\ell}^2 - \frac{1}{6}\right) \frac{\partial^2 K}{\partial \xi^2} + \left(\sum_{\ell=0}^{m} \tfrac{1}{2}\lambda_{m\ell}\, \theta_{m\ell}^2 - \tfrac{1}{2}\right) \frac{\partial^2 K}{\partial x^2} +$$
$$+ \left(\sum_{\ell=0}^{m} \lambda_{m\ell}\nu_{m\ell}\left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K\right) - \frac{1}{3} f'(x_n)\right) \frac{\partial^2 K}{\partial \xi \partial f} +$$
$$+ \left(\sum_{\ell=0}^{m} \tfrac{1}{2}\lambda_{m\ell}\left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i} K\right)^2 - \frac{1}{6}\left(f'(x_n)\right)^2\right) \frac{\partial^2 K}{\partial f^2} +$$

$$+ \left(\sum_{\ell=0}^{m} \lambda_{m\ell}\theta_{m\ell}\left(\mu_\ell F_n'(x_n) + \sum_{i=0}^{m} \lambda_{\ell i}K\right) - \tfrac{1}{2}f'(x_n)\right) \frac{\partial^2 K}{\partial x \partial f} +$$

$$+ \left(\sum_{\ell=0}^{m} \lambda_{m\ell}\theta_{m\ell}\nu_{m\ell} - \tfrac{1}{2}\right) \frac{\partial^2 K}{\partial x \partial \xi}\Bigg] + O(h_n^4).$$

We now use the solutions

$$f(x_n) = F_n(x_n),$$

(3.5) $$f'(x_n) = F_n'(x_n) + K(x_n,x_n,f(x_n)),$$

$$f''(x_n) = F_n''(x_n) + 2\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi} + \frac{\partial K}{\partial f} f'(x_n),$$

to arrive at the following consistency conditions:

(3.6) $$\mu_m = 1 \qquad \rightarrow f_{n+1} = f(x_{n+1}) + O(h_n)$$

(3.7) $$\sum_{\ell=0}^{m} \lambda_{m\ell} = 1 \qquad \rightarrow f_{n+1} = f(x_{n+1}) + O(h_n^2)$$

(3.8) $$\left.\begin{aligned} &\sum_{\ell=0}^{m} \lambda_{m\ell}\nu_{m\ell} = \tfrac{1}{2} \quad \sum_{\ell=0}^{m} \lambda_{m\ell}\theta_{m\ell} = \tfrac{1}{2} \\ &\sum_{\ell=1}^{m} \lambda_{m\ell}\mu_\ell = \tfrac{1}{2} \\ &\sum_{\ell=1}^{m} \lambda_{m\ell} \sum_{i=0}^{m} \lambda_{\ell i} = \tfrac{1}{2} \end{aligned}\right\} \rightarrow f_{n+1} = f(x_{n+1}) + O(h_n^3)$$

$$
\left.
\begin{aligned}
&\sum_{\ell=1}^{m} \lambda_{m\ell}\mu_\ell^2 = \frac{1}{3}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell} \sum_{i=0}^{m} \lambda_{\ell i}\nu_{\ell i} = \frac{1}{6}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell} \sum_{i=1}^{m} \lambda_{\ell i}\mu_i = \frac{1}{6}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell} \sum_{i=0}^{m} \lambda_{\ell i} \sum_{k=0}^{m} \lambda_{ik} = \frac{1}{6}\\
&\sum_{\ell=0}^{m} \lambda_{m\ell}\nu_{m\ell}^2 = \frac{1}{3} \quad \sum_{\ell=0}^{m} \lambda_{m\ell}\theta_{m\ell}^2 = \frac{1}{3}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell}\nu_{m\ell}\mu_\ell = \frac{1}{3}\\
(3.9)\qquad &\sum_{\ell=1}^{m} \lambda_{m\ell}\nu_{m\ell} \sum_{i=0}^{m} \lambda_{\ell i} = \frac{1}{3}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell} \left(\sum_{i=0}^{m} \lambda_{\ell i}\right)^2 = \frac{1}{3}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell}\mu_\ell \sum_{i=0}^{m} \lambda_{\ell i} = \frac{1}{3}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell} \sum_{i=0}^{m} \lambda_{\ell i}\theta_{\ell i} = \frac{1}{3}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell}\theta_{m\ell}\mu_\ell = \frac{1}{2}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell}\theta_{m\ell} \sum_{i=0}^{m} \lambda_{\ell i} = \frac{1}{2}\\
&\sum_{\ell=1}^{m} \lambda_{m\ell}\theta_{m\ell}\nu_{m\ell} = \frac{1}{2}
\end{aligned}
\right\} \rightarrow f_{n+1} = f(x_{n+1}) + O(h_n^4)
$$

3.2 MULTISTEP METHODS

In case of (2.8) consistency conditions are most easily derived by requiring that the sum in (2.8) is an exact approximation to the integral for the functions $K(x_{n+1},x,f(x)) = x^r$, $r = 0,1,\ldots,p-1$. By assuming that $f_j = f(x_j)$ and $\tilde{F}_j(x) = F_j(x)$, $j = 0,1,\ldots,n$, we obtain the conditions for p-th order consistency

$$(3.10) \qquad h_n \sum_{\ell=0}^{k} b_{n\ell}(x_{n+1-\ell} - x_n)^r \quad \int_{x_n}^{x_{n+1}} (x-x_n)^r \, dx, \quad r = 0,1,\ldots,p-1.$$

By introducing the quantities

$$(3.11) \qquad q_{n\ell} = \frac{x_{n-\ell}-x_n}{h_n}, \quad \ell = -1,\ldots,k-1,$$

the equations (3.10) can be compactly written in the form

$$(3.10') \qquad \sum_{\ell=0}^{k} b_{n\ell}\, q_{n\ell-1}^{r} = \frac{1}{r+1}, \quad r = 0,1,\ldots,p-1.$$

These conditions are identical to those of the Adams-Moulton formulas for ordinary differential equations.

4. CONVERGENCE

Let us assume that the approximation $\tilde{F}_n(x)$ satisfies the relation

$$(4.1) \qquad \tilde{F}_n(x_{n+1}) = g(x_{n+1}) + \int_{x_0}^{x_n} K(x_{n+1},\xi,\tilde{f}(\xi))d\xi + E_n(h),$$

where $E_n(h) \to 0$ as $h = \max_j h_j \to 0$. Secondly, we assume that our numerical scheme is a consistent approximation to the integral equation, i.e. scheme (2.3') satisfies the condition

$$(4.2) \qquad \Phi_n(K(x,\xi,f(\xi))) = \int_{x_n}^{x_{n+1}} K(x_{n+1},\xi,f(\xi))d\xi + T_n(h)$$

where $T_n(h) \to 0$ as $h \to 0$. From (2.1) and (2.3') it follows that

$$f(x_{n+1}) - f_{n+1} = F_n(x_{n+1}) - \tilde{F}_n(x_{n+1}) +$$

$$+\int_{x_n}^{x_{n+1}} K(x_{n+1},\xi,f(\xi))d\xi - \Phi_n(K(x,\xi,\tilde{f}(\xi))d\xi$$

$$= \int_{x_0}^{x_n} \left[K(x_{n+1},\xi,f(\xi)) - K(x_{n+1},\xi,\tilde{f}(\xi))\right]d\xi +$$

$$+\int_{x_n}^{x_{n+1}} K(x_{n+1},\xi,f(\xi)) - \Phi_n(K(x,\xi,f(\xi))) +$$

$$+ \Phi_n(K(x,\xi,f(\xi))) - \Phi_n(K(x,\xi,\tilde{f}(\xi)))$$

$$+ E_n(h).$$

Thus

(4.3) $$f(x_{n+1}) - f_{n+1} = \int_{x_0}^{x_n} \left[K(x_{n+1},\xi,f(\xi)) - K(x_{n+1},\xi,\tilde{f}(\xi))\right]d\xi +$$

$$+ \Phi_n(K(x,\xi,f(\xi))) - \Phi_n(K(x,\xi,\tilde{f}(\xi))) +$$

$$- (E_n(h) + T_n(h)).$$

Finally, we assume that K and Φ_n satisfy for $n \geq k-1$ the following conditions:

$$\sum_{j=0}^{n} w_{nj}\, K(x,x_j,\phi(x_j)) = \int_{x_0}^{x_n} K(x,\xi,\phi(\xi))d\xi + O(h^{q+1}),$$

(4.4) $$|K(x,\xi,f) - K(x,\xi,\tilde{f})| \leq L_1|f-\tilde{f}|,$$

$$|\Phi_n(K(x,\xi,f)) - \Phi_n(K(x,\xi,\tilde{f}))| \leq L_2\, h \sum_{\ell=0}^{k} f(x_{n+1-\ell}) - f_{n+1-\ell}|,$$

where L_1 and L_2 are the Lipschitz constants and k is the number of f_j-values used in the formula for Φ_n. The resulting error formula becomes

$$(4.3') \qquad f(x_{n+1}) - f_{n+1} = \sum_{j=0}^{n} w_{nj}[K(x_{n+1},x_j,f(x_j)) - K(x_{n+1},x_j,f_j] +$$
$$+ \Phi_n(K(x,\xi,f(\xi))) - \Phi_n(K(x,\xi,\tilde{f}(\xi))) -$$
$$- (E_n(h) + T_n(h)) + O(h^{q+1}).$$

In order to draw conclusions from this relation we need the following lemma:

<u>LEMMA</u> 4.1. *Let* $\varepsilon_{n+1}, n = 0,1,2\ldots$ *satisfy the conditions*

$$|\varepsilon_{n+1}| \le L \sum_{j=0}^{n} |\varepsilon_j| + M, \; L,\; M > 0$$

then we have

$$|\varepsilon_{n+1}| \le (1+L)^n \, (M + L|\varepsilon_0|).$$

<u>PROOF</u>. See Henrici [12, p. 312].

Let us now consider the case where F_n is approximated by formula (2.2), that implies $E_n(h) = O(h^{q+1})$. Furthermore, let the scheme be p-th order consistent, i.e. $T_n(h) = O(h^{q+1})$. From (4.4) and (4.3') it then follows that

$$(1 - L_2 h)|f(x_{n+1}) - f_{n+1}| \le L_1 \sum_{j=0}^{n} w_{nj}|f(x_j) - f_j| +$$
$$+ L_2\, h \sum_{\ell=1}^{k} |f(x_{n+1-\ell}) - f_{n+1-\ell}| + O(h^{p+1}) + O(h^{q+1}) \le$$
$$\le (L_1 w + L_2) h \sum_{j=0}^{n} |f(x_j) - f_j| + O(h^{p+1}) + O(h^{q+1}), \quad n \ge k-1,$$

where

$$w = \max_{i,j} \frac{|w_{ij}|}{h} \quad .$$

We now apply lemma 4.1 with

$$L = \frac{L_1 w + L_2}{1 - L_2 h}\, h, \quad M = L \sum_{j=0}^{k-2} |f(x_j) - f_j| + O(h^{p+1}) + O(h^{q+1}),$$

to obtain

$$(4.5) \qquad |f(x_{n+1}) - f_{n+1}| \le (1+L)^{n-k+1}(M + L|f(x_{k-1}) - f_{k-1}|) \quad .$$

Assuming that the starting values have errors of order $r = \min(p,q)$ in h, i.e.

$$(4.6) \qquad |f(x_j) - f_j| = O(h^r), \quad j = 0,1,\dots,k-1,$$

we finally obtain from (4.5)

$$(4.7) \qquad f(x_{n+1}) - f_{n+1} = O(h^{p+1}) + O(h^{q+1}) \quad \text{as } h \to 0$$

where x_{n+1} is kept fixed.

When (2.2') is used we have

$$E_n(h) = \tilde{F}_n(x_{n+1}) - g(x_{n+1}) - \int_{x_0}^{x_n} K(x_{n+1},\xi,f(\xi))d\xi =$$

$$= f_n - g(x_n) - \sum_{j=0}^{n} w_{nj} K(x_n,x_j,f_j) +$$

$$+ \sum_{j=0}^{n} w_{nj} K(x_{n+1},x_j,f_j) - \int_{x_0}^{x_n} K(x_{n+1},\xi,\tilde{f}(\xi))d\xi$$

$$= f_n - f(x_n) + \int_{x_0}^{x_n} [K(x_n,\xi,f(\xi)) - K(x_{n+1},\xi,\tilde{f}(\xi))]d\xi +$$

$$+ \sum_{j=0}^{n} w_{nj}[K(x_{n+1},x_j,f_j) - K(x_n,x_j,f_j)] \quad .$$

Substitution into (4.3) yields

$$f(x_{n+1}) - f_{n+1} = f(x_n) - f_n + \int_{x_0}^{x_n} [K(x_{n+1},\xi,f(\xi)) - K(x_n,\xi,f(\xi))]d\xi \tag{4.8}$$
$$+ \sum_{j=0}^{n} w_{nj}[K(x_n,x_j,f_j) - K(x_{n+1},x_j,f_j)] +$$
$$+ \Phi_n(K(x,\xi,f(\xi))) - \Phi_n(K(x,\xi,\tilde{f}(\xi))) - T_n(h).$$

We now define the error function $C_n(x,h)$ by

$$\int_{x_0}^{x_n} K(x,\xi,f(\xi))d\xi = \sum_{j=0}^{n} w_{nj}\, K(x,x_j,f(x_j)) + C_n(x,h)h^{q+1}.$$

Formula (4.8) may be written as

$$f(x_{n+1}) - f_{n+1} = f(x_n) - f_n + \sum_{j=0}^{n} w_{nj}[K(x_{n+1},x_j,f(x_j) +$$
$$- K(x_n,x_j,f(x_j)) + K(x_n,x_j,f_j) - K(x_{n+1},x_j,f_j)] +$$
$$+ \Phi_n(K(x,\xi,f(\xi))) - \Phi_n(K(x,\xi,\tilde{f}(\xi))) - T_n(h) +$$
$$+ [C_n(x_{n+1},h) - C_n(x_n,h)]h^{q+1}.$$

In addition to (4.4) we now also have to require that

$$\begin{aligned} &|C_n(x_{n+1},h) - C_n(x_n,h)| \le L_3 h, \\ &|K(x,\xi,f) - K(x,\xi,\tilde{f}) + K(\tilde{x},\xi,\tilde{f}) - K(\tilde{x},\xi,f)| \le L_4|x-\tilde{x}||f-\tilde{f}|. \end{aligned} \tag{4.9}$$

From (4.8), (4.4) and (4.9) it then follows that for n = k-1

$$(1 - L_2 h)|f(x_{n+1}) - f_{n+1}| \le |f(x_n) - f_n| + L_4 h^2 w \sum_{j=0}^{n} |f(x_j) - f_j| +$$
$$+ L_2 h \sum_{\ell=1}^{k} |f(x_{n+1-\ell}) - f_{n+1-\ell}| + |T_n(h)| + L_3 h^{q+2}.$$

Thus, we find the inequality

$$|f(x_{n+1}) - f_{n+1}| \le |f(x_n) - f_n| + \frac{2L_2}{1 - L_2 h} h \sum_{j=n+1-k}^{n} |f(x_j) - f_j| +$$

$$+ \frac{L_4 w}{1 - L_2 h} h^2 \sum_{j=0}^{n} |f(x_j) - f_j| + \frac{|T_n(h)| + L_3 h^{q+2}}{1 - L_2 h} .$$

A simple calculation yields

$$|f(x_{n+1}) - f_{n+1}| \le (1 - L_2 h)^{-1} \Big\{ \sum_{j=k-1}^{n} \Big[L_4 w h^2 \sum_{i=0}^{j} |f(x_i) - f_i| +$$

$$+ |T_j(h)| + L_3 h^{q+2} \Big] + 2kL_2 h \sum_{j=0}^{n} |f(x_j) - f_j| \Big\} \le$$

$$\le (1 - L_2 h)^{-1} \Big\{ [nL_4 w h^2 + 2kL_2 h] \sum_{j=0}^{n} |f(x_j) - f_j| +$$

$$+ n [\max_{0 \le j \le n} |T_j(h)| + L_3 h^{q+2}] \Big\}.$$

Applying lemma 4.1 with

$$L = \frac{nL_4 w h^2 + 2kL_2 h}{1 - L_2 h} ,$$

$$M = L \sum_{j=0}^{k-2} |f(x_j) - f_j| + n \frac{\max_j |T_j(h)| + L_3 h^{q+2}}{1 - L_2 h}$$

leads to inequality (4.5). Keeping the point x_{n+1} fixed, we have $n = \mathcal{O}(h^{-1})$ so that

$$L = \mathcal{O}(h), \; M = \mathcal{O}(h^{r+1}) + \mathcal{O}(h^{q+1}) + \mathcal{O}(h^p) ,$$

where we have again assumed that the starting values satisfy (4.6). It is now easily seen that (4.5) leads to the result

$$f(x_{n+1}) - f_{n+1} = O(h^p) + O(h^{q+1}) \quad \text{as } h \to 0. \tag{4.10}$$

5. STABILITY

Before studying the stability of scheme (2.4) and (2.8) we consider the stability of the integral equation itself. Consider the variational equation

$$\Delta f(x) = \int_{x_0}^{x} \frac{k}{f} (x,\xi,f(\xi)) \, \Delta f(\xi) d\xi \tag{5.1}$$

From this relation it follows that

$$\begin{aligned}\Delta f(x_{n+1}) = \Delta f(x_n) &+ \int_{x_0}^{x_n} [\frac{\partial K}{\partial f} (x_{n+1},\xi,f(\xi)) + \\ &- \frac{\partial K}{\partial f} (x_n,\xi,f(\xi))] \, \Delta f(\xi) d\xi + \\ &+ \int_{x_n}^{x_{n+1}} \frac{\partial K}{\partial f} (x_{n+1},\xi,f(\xi)) \, \Delta f(\xi) d\xi.\end{aligned}$$

Let us define the quantity

$$\Delta G(x_n) = \int_{x_0}^{x_n} \frac{\partial^2 K}{\partial x \partial f} (x_n,\xi,f(\xi)) \, \Delta f(\xi) d\xi$$

and assume that $\partial^2 K/\partial x \partial f$ is a slowly varying function of x, and $\partial K/\partial f$ a slowly varying function of ξ and f. Then, we may write

$$\begin{aligned}\Delta f(x_{n+1}) \cong \Delta f(x_n) &+ h_n \, \Delta G(x_n) + [\frac{\partial K}{\partial f} (x_n,x_n,f(x_n)) + \\ &+ h_n \frac{\partial^2 K}{\partial x \partial f} (x_n,x_n,f(x_n)) \int_{x_n}^{x_{n+1}} \Delta f(\xi) d\xi.\end{aligned}$$

$$\Delta G(x_{n+1}) \cong \Delta G(x_n) + \frac{\partial^2 K}{\partial x \partial f} (x_n,x_n f(x_n)) \int_{x_n}^{x_{n+1}} \Delta f(\xi) d\xi.$$

Using the abbreviations

$$J(x_n) = \frac{\partial K}{\partial f}(x_n,x_n,f(x_n)),$$
$$H(x_n) = \frac{\partial^2 K}{\partial x \partial f}(x_n,x_n,f(x_n))$$

and approximating the integral with the trapezoidal rule, we obtain

(5.2) $$\begin{pmatrix} 1-\frac{1}{2}h_n(J(x_n) + h_n H(x_n)) & 0 \\ -\frac{1}{2}h_n H(x_n) & 1 \end{pmatrix} \begin{pmatrix} \Delta f(x_{n+1}) \\ \Delta G(x_{n+1}) \end{pmatrix} \cong$$

$$= \begin{pmatrix} 1+\frac{1}{2}h_n(J(x_n) + h_n H(x_n)) & h_n \\ \frac{1}{2}h_n H(x_n) & 1 \end{pmatrix} \begin{pmatrix} \Delta f(x_n) \\ \Delta G(x_n) \end{pmatrix} + O(h_n^3).$$

Neglecting the $O(h_n^3)$ terms we may conclude that this error equation is stable when its characteristic equation has its eigenvalues within or on the unit circle. A simple calculation yields

$$[1-\tfrac{1}{2}h_n(J(x_n) + h_n H(x_n))]\zeta^2 - [2+\tfrac{1}{2}h_n^3 H(x_n)]\zeta +$$
$$+ 1 + \tfrac{1}{2}h_n J(x_n) = 0.$$

It is easily verified (cf. the analyis of equation (5.22) in section 5.2) that this equation has its roots within or on the unit circle when

$$H(x_n) \le 0, \frac{J(x_n)}{H(x_n)} \ge -\tfrac{1}{2}h.$$

Thus, for $h \to 0$ the stability conditions become

(5.3) $$\frac{\partial K}{\partial f}(x_n,x_n,f(x_n)) \le 0, \frac{\partial^2 K}{\partial x \partial f}(x_n,x_n,f(x_n)) \le 0.$$

In case of *strict* inequality we shall speak of *strong stability*.

5.1 STABILITY ANALYSIS OF SINGLE-STEP METHODS

Let us perturb the numerical values f_j, $j = 0,1,\dots,n$ by perturbations Δf_j and denote the amount by which $f_{n+1}^{(\ell)}$ and $\tilde{F}_n(x)$ are perturbed by $\Delta f_{n+1}^{(\ell)}$ and $\Delta\tilde{F}_n(x)$, respectively. The perturbation of f_{n+1} is then approximately determined by the scheme

$$\Delta f_{n+1}^{(0)} = \Delta f_n,$$

(5.4) $$\Delta f_{n+1}^{(j)} = \Delta\tilde{F}_n(x_n+\mu_j h_n) + h_n \sum_{\ell=0}^{m} \lambda_{j\ell} \frac{\partial K}{\partial f}(x_n+\theta_{j\ell}h_n, x_n+\nu_{j\ell}h_n, f_{n+1}^{(\ell)})\ \Delta f_{n+1}^{(\ell)},$$

$$\Delta f_{n+1} = \Delta f_{n+1}^{(m)}, \qquad j = 1,2,\dots,m,$$

provided that the perturbations Δf_j are sufficiently small. In most studies of stability the considerations are restricted to the model equation (cf. KOBAYASI [6], LINZ [8] or NOBLE [9])

(5.5) $$f(x) = 1 - a\int_0^x f(\xi)d\xi,$$

that it is assumed that the kernel function $K(x,\xi,f)$ satisfies the conditions.

$$\frac{\partial K}{\partial x} = \text{constant}, \quad \frac{\partial K}{\partial x} = \frac{\partial K}{\partial \xi} = \frac{dg}{dx} = 0.$$

Instead of these rigorous restrictions to the class of integral equations to be analyzed, we prefer to state in more details what restrictions are needed to give a stability analysis. When formula (2.2) is used for the evaluation of $\tilde{F}_n(x)$, we require the following properties of the kernel function $K(x,\xi,f)$ (K is assumed to be sufficiently differentiable):

(5.6)
$$\frac{\partial K}{\partial f}(x,\xi,f) = \frac{\partial K}{\partial f}(x,x_n,f_n) \quad \text{for} \quad (\xi,f) \in U_\varepsilon(x_n,f_n)$$

$$\frac{\partial^2 K}{\partial x\partial f}(x,x_j,f_j) \cong \frac{\partial^2 K}{\partial x\partial f}(x_n,x_j,f_j)\Big|, \quad j = 0,1,\dots,n \quad \text{for } x \in U_\varepsilon(x_n)$$

where $U_\varepsilon(\cdot)$ denotes a small neighbourhood of $(\cdot)$. When these conditions are satisfied $\Delta\tilde{F}_n(x)$ is approximated by

$$(5.7)\qquad \Delta\tilde{F}_n(x) \cong \sum_{j=0}^{n} w_{nj} \frac{\partial K}{\partial f}(x,x_j,f_j)\,\Delta f_j =$$

$$= \sum_{j=0}^{m} w_{nj}\left[\frac{\partial K}{\partial f}(x_n,x_j,f_j) + (x-x_n)\frac{\partial^2 K}{\partial x\partial f}(\bar{x},x_j,f_j)\right]\Delta f_j \cong$$

$$\cong \Delta\tilde{F}_n(x_n) + (x-x_n)\,\Delta G_n,$$

where

$$\Delta G_n = \sum_{j=0}^{n} w_{nj}\frac{\partial^2 K}{\partial x\partial f}(x_n,x_j,f_j)\,\Delta f_j .$$

By writing

$$\frac{\partial^2 K}{\partial x\partial f}(x_n,x_n,f_n) = H_n,$$

$$\frac{\partial K}{\partial f}(x_n+\theta_{j\ell}h_n, x_n+\nu_{j\ell}h_n, f_{n+1}^{(\ell)}) = J_n + \theta_{j\ell}h_nH_n,$$

scheme (5.4) reduces to

$$\Delta f_{n+1}^{(0)} = \Delta f_n,$$

$$(5.4')\qquad \Delta f_{n+1}^{(j+1)} \cong \tilde{F}_n(x_n) + \mu_j h_n \Delta G_n + \sum_{\ell=0}^{m} \lambda_{j\ell}[h_nJ_n + \theta_{j\ell}h_n^2H_n]\,\Delta f_{n+1}^{(\ell)},$$

$$\Delta f_{n+1} \cong \Delta f_{n+1}^{(m)},$$

These formulas suggest to express $\Delta f_{n+1}^{(j)}$ in the form

$$(5.4'')\qquad \Delta f_{n+1}^{(j)} \cong Q_j\,\Delta f_n + R_j\,\Delta\tilde{F}_n(x_n) + S_j h_n\,\Delta G_n,$$

where Q_j, R_j and S_j are polynomials or rational functions in the arguments h_nJ_n and $h_n^2H_n$. By substituting (5.4") into (5.4') we find ($z=h_nJ_n, y = h_n^2H_n$)

$$Q_0(z,y) = 1,\quad Q_j(z,y) = \sum_{\ell=0}^{m} \lambda_{j\ell}(z+\theta_{j,\ell}y)\,Q_\ell(z,y),$$

$$R_0(z,y) = 0,\quad R_j(z,y) = 1 + \sum_{\ell=0}^{m} \lambda_{j\ell}(z+\theta_{j,\ell}y)\,R_\ell(z,y),$$

$$S_0(z,y) = 0,\ S_j(z,y) = \mu_j + \sum_{\ell=0}^{m} \lambda_{j\ell}(z+\theta_{j,\ell}y)\ S_\ell(z,y),$$

from which the functions Q_m, R_m and S_m, to be called *stability functions* in this paper, can be derived. Thus,

(5.8) $$\Delta f_{n+1} = Q_m(h_nJ_n, h_n^2H_n)\ \Delta f_n + R_m(h_nJ_n, h_n^2H_n)\ \Delta\tilde{F}_n(x_n) + S_m(h_nJ_n, h_n^2H_n)h_n\Delta G_n.$$

(Note that R_m and S_m are identical when for all j, $\mu_j = 1$.)
Furthermore, from the relation

$$\tilde{F}_{n+1}(x_{n+1}) - \tilde{F}_n(x_n) = g(x_{n+1}) - g(x_n) + \sum_{j=0}^{n+1} w_{n+1,j}\ K(x_{n+1},x_j,f_j) + - \sum_{j=0}^{n} w_{nj}\ K(x_n,x_j,f_j)$$

we find, using conditions (5.6),

$$\Delta\tilde{F}_{n+1}(x_{n+1}) \cong \Delta\tilde{F}_n(x_n) + \sum_{j=0}^{n} (w_{n+1j}-w_{nj})\ \frac{\partial K}{\partial f}\ (x_n,x_j,f_j)\ \Delta f_j + + h_n \sum_{j=0}^{n+1} w_{n+1j}\ \frac{\partial^2 K}{\partial x\partial f}\ (x_n,x_j,f_j)\ \Delta f_j + + w_{n+1n+1}\ \frac{\partial K}{\partial f}\ (x_n,x_n,f_n)\ \Delta f_{n+1},$$

or, when $\bar{n}$ is close to n,

(5.9) $$\Delta\tilde{F}_{n+1}(x_{n+1}) \cong \Delta\tilde{F}_n(x_n) + \sum_{j=\bar{n}}^{n} (w_{n+1j} - w_{nj})J_n\ \Delta f_j + + h_n\ \Delta\ G_{n+1} + w_{n+1n+1}J_n\ \Delta\ f_{n+1},$$

$w_{n+1\bar{n}}$ being the first weight in the row w_{n+1j} which differs from w_{nj}.
Finally, we have from the definition of ΔG_n the relation

(5.10) $$\Delta G_{n+1} = \Delta G_n + \sum_{j=\bar{n}}^{n} (w_{n+1j} - w_{nj})\ H_n\ \Delta f_j + w_{n+1n+1}\ H_n\ \Delta f_{n+1}.$$

Introducing the vectors

$$\Delta\vec{V}_n = (\Delta f_n, \Delta f_{n-1}, \ldots, \Delta f_{n+1}, \Delta f_{\bar{n}}, \Delta\tilde{F}_n(x_n), \Delta G_n)^T$$

we arrive at the relation

$$A_n \Delta\vec{V}_{n+1} = B_n \Delta\vec{V}_n, \tag{5.11}$$

where A_n and B_n are (square) matrices defined by

$$A_n = \begin{pmatrix} 1 & 0 & & & & 0 \\ 0 & 1 & & & & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 & 0 \\ -w_{n+1,n+1} J_n & 0 & \ldots & 0 & 1 & -h_n \\ -w_{n+1,n+1} H_n & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}, \quad \Delta w_j = w_{n+1\,j} - w_{nj},$$

$$B_n = \begin{pmatrix} Q_m & 0 & \ldots & 0 & R_m & h_n S_m \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 & 0 \\ \Delta w_n J_n & \Delta w_{n-1} J_n & \ldots & \Delta w_{\bar{n}} J_n & 1 & 0 \\ \Delta w_n H_n & \Delta w_{n-1} H_n & \ldots & \Delta w_{\bar{n}} H_n & 0 & 1 \end{pmatrix}.$$

The vector of perturbations $\Delta\vec{V}_n$ remains bounded in some norm $\|\cdot\|$ when

$$\| A_n^{-1} B_n \| \leq 1. \tag{5.12}$$

A necessary condition to satisfy this inequality is the requirement that all eigenvalues ζ of $A_n^{-1} B_n$ are within or on the unit circle, i.e. the conditions that the roots of the characteristic equation

$$\det(B_n - \zeta A_n) = 0 \tag{5.13}$$

are within or on the unit circle. Note that the degree of this equation can be kept low by choosing the weights w_{nj} such that $\bar{n}$ is close to n. In general, this implies uniform step sizes h_n.

In order to illustrate the preceding results we derive the characteristic equation for the cases where $\tilde{F}_n(x)$ is estimated by the Trapezoidal rule and Simpson's rule (+ 3/8-rule) using uniform integration steps.

Trapezoidal rule + m-point Runge-Kutta

In this case we have (i = 1,2,...n+1 and j = 0,1,...,n+1)

$$(wij) = h \begin{pmatrix} 1/2 & 1/2 & 0 & \dots & & 0 \\ 1/2 & 1 & 1/2 & & & \\ 1/2 & 1 & 1 & & & \\ \vdots & \vdots & & \ddots & & \vdots \\ 1/2 & 1 & \dots & 1 & 1/2 & 0 \\ 1/2 & 1 & \dots & 1 & 1 & 1/2 \end{pmatrix}_{(n+1)*(n+2)}$$

so that the value of $\bar{n}$ in relations (5.9) and (5.10) equals n. Thus, formula (5.11) becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ -\tfrac{1}{2}hJ_n & 1 & -h \\ -\tfrac{1}{2}hH_n & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta f_{n+1} \\ \Delta\tilde{F}_{n+1}(x_{n+1}) \\ \Delta G_{n+1} \end{pmatrix} =$$

$$= \begin{pmatrix} Q_m & R_m & h_n S_m \\ \tfrac{1}{2}hJ_n & 1 & 0 \\ \tfrac{1}{2}hH_n & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta f_n \\ \Delta\tilde{F}_n(x_n) \\ \Delta G_n \end{pmatrix}$$

yielding the characteristic equation ($z = hJ_n$, $y = h^2H_n$)

$$(5.15)\qquad \zeta^3 - [2+Q_m(z,y) + \tfrac{1}{2}zR_m(z,y) + \tfrac{1}{2}y(R_m(z,y) + S_m(z,y)]\zeta^2 +$$
$$+ [1+2Q_m(z,y) - \tfrac{1}{2}yR_m(z,y)]\zeta +$$
$$- [Q_m(z,y) - \tfrac{1}{2}zR_m(z,y) - \tfrac{1}{2}yS_m(z,y)] = 0.$$

In the particular case where $|h^2 H_n|$ is very small, this equation may be written as

$$(5.15') \qquad (1-\zeta)[\zeta^2 - [1+Q_m(z.0) + \tfrac{1}{2}zR_m(z,0)]\zeta + Q_m(z,0) - \tfrac{1}{2}zR_m(z,0)] = 0.$$

The roots are within or on the unit circle when

$$(5.16) \qquad \begin{aligned} zR_m(z,0) &\le 0, \\ Q_m(z,0) &\ge -1, \\ Q_m(z,0) - \tfrac{1}{2}zR_m(z,0) &\le 1. \end{aligned}$$

These inequalities determine the interval of stability $-\beta \le z \le 0$ and the corresponding stability condition (strict inequality corresponds to *strong* stability)

$$(5.17) \qquad h \le \frac{\beta}{|J_n|}, \quad J_n \le 0.$$

Application of these criteria to a number of Runge-Kutta methods may be found in [11].

Simpson's Rule + m-point Runge-Kutta

Leaving aside the starting procedure, Simpson's Rule provides the weights w_{nj} when n is even and together with the 3/8-rule it provides the weights w_{nj} when n is odd; thus

$$(5.18) \qquad (w_{ij}) = h \begin{pmatrix} 1/3 & 4/3 & 1/3 & & & & & & & & & \\ 3/8 & 9/8 & 9/8 & 3/8 & & & & & & & & \\ 1/3 & 4/3 & 2/3 & 4/3 & 1/3 & & & & & & & \\ 1/3 & 4/3 & 17/24 & 9/8 & 9/8 & 3/8 & & & & & & \bigcirc \\ 1/3 & 4/3 & 2/3 & 4/3 & 2/3 & 4/3 & 1/3 & & & & & \\ 1/3 & 4/3 & 2/3 & 4/3 & 17/24 & 9/8 & 9/8 & 3/8 & & & & \\ 1/3 & 4/3 & & \vdots & & & 4/3 & 2/3 & 4/3 & 1/3 & & \\ 1/3 & 4/3 & & \cdots & & & 4/3 & 17/24 & 9/8 & 9/8 & 3/8 & \\ 1/3 & 4/3 & & \cdots & & & 4/3 & 2/3 & 4/3 & 2/3 & 4/3 & 1/3 \end{pmatrix}$$

For odd values of n relation (5.11) becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{3}hJ_n & 0 & 0 & 0 & 1 & -h \\ -\frac{1}{3}hH_n & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta f_{n+1} \\ \Delta f_n \\ \Delta f_{n-1} \\ \Delta f_{n-2} \\ \Delta\tilde{F}_{n+1}(x_{n+1}) \\ \Delta G_{n+1} \end{pmatrix} =$$

$$= \begin{pmatrix} Q_m & 0 & 0 & 0 & R_m & hS_m \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{23}{24}hJ_n & -\frac{11}{24}hJ_n & \frac{5}{24}hJ_n & -\frac{1}{24}hJ_n & 1 & 0 \\ \frac{23}{24}hH_n & -\frac{11}{24}hJ_n & \frac{5}{24}hH_n & -\frac{1}{24}hH_n & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta f_n \\ \Delta f_{n-1} \\ \Delta f_{n-2} \\ \Delta f_{n-3} \\ \Delta\tilde{F}_n(x_n) \\ \Delta G_n \end{pmatrix},$$

which has the characteristic equation

(5.19) $$\det \begin{bmatrix} Q_m-\zeta & 0 & 0 & 0 & R_m & S_m \\ 1 & -\zeta & 0 & 0 & 0 & 0 \\ 0 & 1 & -\zeta & 0 & 0 & 0 \\ 0 & 0 & 1 & -\zeta & 0 & 0 \\ (23+8\zeta)z & -11z & 5z & -z & 24(1-\zeta) & 24\zeta \\ (23+8\zeta)y & -11y & 5y & -y & 0 & 24(1-\zeta) \end{bmatrix} = 0,$$

where we have again written $z = hJ_n$ and $y = h^2H_n$. Putting $y = 0$ (small values of $|h^2H_n|$) this equation reduces to a fifth degree polynomial given by

(5.19a') $$\zeta^5 - [1+Q_m(z,0) + \frac{1}{3}zR(z,0)]\,\zeta^4 - [\frac{23}{24}zR_m(z,0) - Q_m(z,0)]\zeta^3 + \\ + \frac{11}{24}zR_m(z,0)\zeta^2 - \frac{5}{24}zR_m(z,0)\zeta + \frac{1}{24}zR_m(z,0) = 0.$$

For even values of n we find in a similar way the fourth degree polynomial

(5.19b') $$\zeta^4 - [1+Q_m(z,0) + \frac{3}{8} z R_m(z,0)]\,\zeta^3 - [\frac{19}{24} z R_m(z,0) - Q_m(z,0)]\zeta^2 + \frac{5}{24} z R_m(z,0)\zeta - \frac{1}{24} z R_m(z,0) = 0.$$

Application of these equations to several Runge-Kutta formulas may be found in RECKERS [11].

5.2 MULTISTEP METHODS

From (2.8) and (5.7) it follows that

(5.20) $$\Delta f_{n+1} \cong \Delta\tilde{F}_n(x_{n+1}) + h_n \sum_{\ell=0}^{k} b_{n,\ell} \frac{\partial K}{\partial f}(x_{n+1}, x_{n+1-\ell}, f_{n+1-\ell})\, \Delta f_{n+1-\ell} \cong$$
$$\cong \Delta\tilde{F}_n(x_n) + h_n\, \Delta G_n + h_n \sum_{\ell=0}^{k} b_{n,\ell}[J_n + h_n H_n]\, \Delta f_{n+1-\ell}.$$

Together with the relations (5.9) and (5.10) this formula describes the stability of the process. Let us assume that the number $\bar{n}$ occuring in (5.9) and (5.10) satisfies the inequality

$$\bar{n} \geq n-k+1$$

we then may introduce the vectors

$$\Delta\vec{V}_n = (\Delta f_n, \Delta f_{n-1}, \ldots, \Delta f_{\bar{n}}, \ldots, \Delta f_{n+1-k}, \Delta\tilde{F}_n(x_n), \Delta G_n)^T$$

and obtain the relation (cf.(5.11))

(5.21) $$A_n \Delta\vec{V}_{n+1} = B_n \Delta\vec{V}_n,$$

where

$$A_n = \begin{pmatrix} 1-h_n b_{n,0} J_{n+1} & & & & & & & \\ 0 & 1 & & & & & \bigcirc & \\ 0 & 0 & 1 & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & & \cdot & & & & \\ \cdot & \cdot & & & \cdot & & & \\ 0 & 0 & & & 0 & 1 & 0 & 0 \\ -w_{n+1,n+1} J_n & 0 & \cdots & & 0 & 0 & 1 & -h_n \\ -w_{n+1,n+1} H_n & 0 & \cdots & & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_n = \begin{pmatrix} h_n b_{n1} J_{n+1} & h_n b_{n2} J_{n+1} & \cdots & h_{n,} b_{n-\bar{n}+1} J_{n+1} & \cdots & h_n b_{nk} J_{n+1} & h_n \\ 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & \cdots & 0 & 0 \\ \cdot & & \cdot & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ 0 & 0 & \cdots & 0 & 0 \ldots 0\ 1\ 0 & 0 & 0 \\ \Delta w_n J_n & \Delta w_{n-1} J_n & \cdots & \Delta w_{\bar{n}} J_n & 0 \ldots 0\ 0\ 0 & 1 & 0 \\ \Delta w_n H_n & \Delta w_{n-1} H_n & \cdots & \Delta w_{\bar{n}} H_n & 0 \ldots 0\ 0\ 0 & 0 & 1 \end{pmatrix}$$

and where $J_{n+1} = J_n + h_n H_n$.
The characteristic equation of relation (5.21) is again of the form (cf. (5.13))

$$\det(B_n - \zeta A_n) = 0.$$

Relation (5.21) can be simplified when the multistep method is of the special (usual) form (2.20). We then may write

$$\Delta f_{n+1} \cong \Delta f_n + \sum_0^n (w_{n+1,j} - w_{n,j}) \frac{\partial K}{\partial f} (x_n, x_j, f_j)\, \Delta f_j +$$

$$+ w_{n+1,n+1} \frac{\partial K}{\partial f} (x_n, x_{n+1}, f_{n+1})\, \Delta f_{n+1} +$$

$$+ h_n \sum_0^{n+1} w_{n+1j} \frac{\partial^2 K}{\partial x \partial f} (x_n, x_j, f_j)\, \Delta f_j$$

$$\cong \Delta f_n + \sum_{\bar{n}}^{n} \Delta w_j J_n \Delta f_j + w_{n+1n+1} J_n \Delta f_{n+1} + h_n \Delta G_{n+1}.$$

Together with (5.10) we now have the error equation

$$(5.21') \qquad A_n \Delta\vec{V}_{n+1} = B_n \vec{V}_n$$

where

$$\Delta\vec{V}_n = (\Delta f_n, \Delta f_{n-1}, \ldots, \Delta f_{n+1-k}, \Delta G_n)^T$$

$$A_n = \begin{pmatrix} 1 - w_{n+1n+1} J_n & 0 & & -h_n \\ 0 & 1 & & 0 \\ \cdot & & & \\ \cdot & & \ddots & \\ \cdot & & & \\ -w_{n+1n+1} H_n & 0 & & 1 \end{pmatrix}$$

and

$$B_n = \begin{pmatrix} 1 + \Delta w_n J_n & \Delta w_{n-1} J_n & \cdots & \Delta w_{\bar{n}} J_n & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & 0 & 0 \\ \Delta w_n H_n & \Delta w_{n-1} H_n & \cdots & \Delta w_{\bar{n}} H_n & 1 \end{pmatrix}$$

We shall illustrate these results by deriving the characteristic equations of the trapezoidal rule and of Simpson's (+3/8) rule.

Trapezoidal rule

Let the weights w_{ij} be given by (5.14) and let the parameters $b_{n\ell}$ satisfy relation (2.9). According to (5.21') we have

$$\begin{pmatrix} 1-\frac{1}{2}hJ_n & -h \\ -\frac{1}{2}hH_n & 1 \end{pmatrix} \begin{pmatrix} \Delta f_{n+1} \\ \Delta G_{n+1} \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}hJ_n & 0 \\ \frac{1}{2}hH_n & 1 \end{pmatrix} \begin{pmatrix} \Delta f_n \\ \Delta G_n \end{pmatrix}$$

leading to the characteristic equation

$$\zeta\,[(1-\tfrac{1}{2}z-\tfrac{1}{2}y)\zeta^2 - (\tfrac{1}{2}y+2)\zeta + (1+\tfrac{1}{2}z)] = 0. \tag{5.22}$$

The roots of this equation are within or on the unit circle when

$$\begin{aligned} &\frac{2+z}{2-z-y} \le 1, \\ &\left|\frac{y+4}{2-z-y}\right| \le \frac{4-y}{2-z-y}\,. \end{aligned} \tag{5.23}$$

In figure 5.1 the region of points (z,y) is shown which satisfy these inequalities.

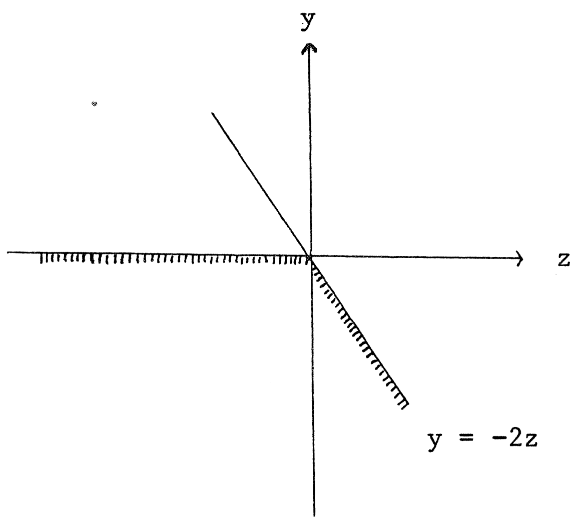

Fig.5.1 Stability region for the trapezoidal rule

Simpson' rule

Let the weights w_{ij} be given by (5.18) and the $b_{n\ell}$ by (2.9). For odd values of n we obtain from (5.21') the relation

$$\begin{pmatrix} 1-\frac{1}{3}hJ_n & 0 & 0 & 0 & -h \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{3}hH_n & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta f_{n+1} \\ \Delta f_n \\ \Delta f_{n-1} \\ \Delta f_{n-2} \\ \Delta G_{n+1} \end{pmatrix} =$$

$$= \begin{pmatrix} 1+\frac{23}{24}hJ_n & -\frac{11}{24}hJ_n & \frac{5}{24}hJ_n & -\frac{1}{24}hJ_n & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{23}{24}hH_n & -\frac{11}{24}hH_n & \frac{5}{24}hH_n & -\frac{1}{24}hH_n & 1 \end{pmatrix} \begin{pmatrix} \Delta f_n \\ \Delta f_{n-1} \\ \Delta f_{n-2} \\ \Delta f_{n-3} \\ \Delta G_n \end{pmatrix},$$

with the characteristic equation

$$\text{(5.24a)} \qquad 24\zeta^3(1-\zeta)^2 + (z-z\zeta-y\zeta)(1-5\zeta+11\zeta^2+23\zeta^3+8\zeta^4) = 0.$$

Similarly, we find for even values of n the equation

$$\text{(5.24b)} \qquad 24\zeta^2(1-\zeta)^2 + (z-z\zeta-y\zeta)(1-5\zeta+\zeta 19\zeta^2+9\zeta^3) = 0.$$

5.3 STABILITY ANALYSIS OF THE MODIFIED FORMULAS

In the preceding analysis it was assumed that $\tilde{F}_n(x)$ is evaluated by formula (2.2). We now study the stability problem when formula (2.2') is used. Again assuming that the kernelfunction satisfies conditions (5.6), we now have for $\Delta\tilde{F}_n(x)$ the relation

$$(5.7')\qquad \Delta\tilde{F}_n(x) \cong \Delta f_n + \sum_{j=0}^{n} w_{nj}\left[\frac{\partial K}{\partial f}(x,x_j,f_j) - \frac{\partial K}{\partial f}(x_n,x_j,f_j)\right]\Delta f_j$$

$$\cong \Delta f_n + (x-x_n)\,\Delta G_n.$$

Single step methods

Substitution of (5,7') into (5.8) yields

$$(5.25)\qquad \Delta f_{n+1} = [Q_m(h_nJ_n,h_n^2H_n) + R_m(h_nJ_n,h_n^2H_n)]\,\Delta f_n +$$

$$+ S_m(h_nJ_n,h_n^2H_n)h_n\,\Delta G_n$$

Together with (5.10) we arrive at the relation

$$(5.26)\qquad A_n\Delta\vec{V}_{n+1} = B_n\Delta\vec{V}_n,$$

where

$$\Delta\vec{V}_n = (\Delta f_n,\ldots,\Delta f_{\bar{n}},\Delta G_n)^T$$

$$A_n = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ -w_{n+1,n+1}H_n & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and

$$B_n = \begin{pmatrix} Q_m+R_m & 0 & & \cdots & 0 & h_nS_m \\ 1 & 0 & & \cdots & 0 & 0 \\ 0 & 1 & & \cdots & 0 & 0 \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \Delta w_n H_n & \Delta w_{n-1} H_n & \cdots & \Delta w_{\bar{n}-1} H_n & \Delta w_{\bar{n}} H_n & 1 \end{pmatrix}.$$

Note that for vanishing H_n equation (5.26) does not depend on the quadrature formula used in the evaluation of $\tilde{F}_n(x)$. In fact, the error equation reduces to that of the equivalent Runge-Kutta formula for ordinary differential equations (cf. (2.4")).

We shall illustrate the application of this error equation by deriving the stability region of the formula defined by the matrix (5.14) (trapezoidal rule). Since we then simply have $\bar{n} = n$ and $\Delta w_n = h/2$, we obtain

$$(5.27) \qquad \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}hH_n & 1 \end{pmatrix} \begin{pmatrix} \Delta f_{n+1} \\ \Delta G_{n+1} \end{pmatrix} = \begin{pmatrix} Q_m+R_m & hS_m \\ \frac{1}{2}hH_n & 1 \end{pmatrix} \begin{pmatrix} \Delta f_n \\ \Delta G_n \end{pmatrix} .$$

The characteristic equation is given by

$$\zeta^2 - [1 + Q_m(z,y) + R_m(z,y) + \tfrac{1}{2}yS_m(z,y)]\zeta + $$
$$+ Q_m(z,y) + R_m(z,y) - \tfrac{1}{2}yS_m(z,y) = 0 .$$

In the (z,y)-plane the stability region, i.e. the set of points (z,y) where $|\zeta(z,y)| < 1$, is given by the inequalities

$$(5.28) \qquad \begin{gathered} -1 \le Q_m(z,y) + R_m(z,y) \le 1 + \tfrac{1}{2}yS_m(z,y) \\ yS_m(z,y) \le 0 \end{gathered}$$

Multistep methods

Substitution of (5.7') into (5.20) yields

$$\Delta f_{n+1} = \Delta f_n + h_n \Delta G_n + h_n \sum_{\ell=0}^{k} b_{n.\ell} \, [J_n + h_n H_n] \, \Delta f_{n+1-\ell}. \tag{5.29}$$

Together with (5.10) this yields the error equation ($\bar{n} \geq n-k+1$)

$$A_n \Delta\vec{V}_{n+1} = B_n \Delta\vec{V}_n, \tag{5.30}$$

where

$$\Delta\vec{V}_n = (\Delta f_n, \Delta f_{n+1}, \ldots, \Delta f_{\bar{n}}, \ldots, \Delta f_{n+1-k}, \Delta G_n)^T,$$

$$A_n = \begin{pmatrix} 1-h_n b_{n0} J_n & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ \vdots & \vdots & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -w_{n+1\,n+1} H_n & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

and

$$B_n = \begin{pmatrix} 1+h_n b_{n1} J_{n+1} & h_n b_{n2} J_{n+1} & \cdots & & h_n b_{nk} J_{n+1} & h_n \\ 1 & 0 & & & 0 & 0 \\ 0 & 1 & & & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 \ldots 1 & 0 & 0 \\ \Delta w_n H_n & \Delta w_{n-1} H_n \cdots & \Delta w_{\bar{n}} H_n & 0 \ldots 0 & 0 & 1 \end{pmatrix},$$

J_{n+1} being again $J_n + h_n H_n$. From this relation the characteristic equation is easily derived.

REFERENCES

[1] BELTJUKOV, B.A., *An analogue of the Runge-Kutta method for the solution of a nonlinear integral equation of the Volterra type*. Differential Equations 1, 417-433 (1965).

[2] DAY, J.T., *On the numerical solution of Volterra integral equations*. BIT 8, 134-137 (1968).

[3] HOOG, DE, F. and R. WEISS, *Implicit Runge-Kutta methods for second kind Volterra integral equaizons*, Numer. Math. 23, 199-213 (1975).

[4] GAREY, L., *Predictor-corrector methods for nonlinear Volterra integral equations of the second kind*, BIT 12, 325-333 (1972).

[5] GAREY, L., *Implicit methods for Volterra integral equations of the second kind*, Proc. Second Manitoba Conference on Numerical Math., 167-177 (1972).

[6] KOBAYASI, M., *On the numerical solution of Volterra integral equations of the second kind by linear multistep methods*. Rep. Stat. Appl. Res. JUSE 13, 1-21 (1966).

[7] LAUDET, M. and H. OULES, *Sur l' integration numerique des equations integrales du type de Volterra. Symposium on the numerical treatment of ordinary differential equations, integral and integro-differential equations*, 117-121. Basel: Birkhauser Verlag (1960).

[8] LINZ, P., *The numerical solution of Volterra integral equations by finite difference methods*. M.R.C. Tech. Summary Report ≠ 825 (1967).

[9] NOBLE, B., *Instability when solving Volterra integral equations of the second kind by multistep methods*. Lecture Notes in Mathematics 109, 23-39 Springer (1969).

[10] POUZET, P., *Method d'integration numerique des equations integrales et integro-differentielles du type de Volterra de seconde espece. Formule du Runge-Kutta. Symposium on the numerical treatment of ordinary differential equations, integral and integro-differential equations,* 362-368. Basel: Birkhauser Verlag (1960).

[11] RECKERS, F.Y., *Stability of Runge-Kutta methods for Volterra integral equations of the second kind,* Technical Note, Mathematisch Centrum, Amsterdam (1977).

[12] HENRICI, P., *Discrete variable methods in ordinary differential equations,* Wiley & Sons, New York (1962).